FROBENIUS RING HOMOMORPHISMS

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Abstract. We introduce the notion of Frobenius ring homomorphisms and show that if $R \to A$ is a Frobenius ring homomorphism then A inherits various homological properties from R. Especially, for a Frobenius ring homomorphism $R \to A$ we show that if R is an Auslander-Gorenstein ring then so is A with inj dim $A \leq$ inj dim R.

1. Preliminaries

Let R be a ring. We denote by Mod-R the category of right R-modules and by mod-R the full subcategory of Mod-R consisting of finitely presented modules. We denote by $\mathcal{P}_R$ the full subcategory of mod-R consisting of projective modules. We denote by R^{op} the opposite ring of R and consider left R-modules as right R^{op}-modules. In particular, we denote by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}) and by gl dim R (resp., gl dim R^{op}) the right (resp., left) global dimension of R. Similarly, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of the right (resp., left) R-module R. Sometimes, we use the notation X_R (resp., ${}_RX$) to stress that the module X considered is a right (resp., left) R-module. For each complex $X^\bullet$ we denote by $\mathrm{Z}^i(X^\bullet)$, $\mathrm{Z}'^i(X^\bullet)$ and $\mathrm{H}^i(X^\bullet)$ the ith cycle, ith cocycle and the ith cohomology, respectively. We denote by $\mathrm{Hom}^\bullet(-,-)$ (resp., $-\otimes^\bullet -$) the hom (resp., tensor) complex. Finally, for a module $X \in$ Mod-R we denote by add(X) the full subcategory of Mod-A consisting of direct summands of finite direct sums of copies of X.

In this section, we recall several basic facts which are well-known.

Proposition 1 (Auslander). *Let R be a left and right noetherian ring. Then for any $n \geq 0$ the following are equivalent:*

1. *In a minimal injective resolution $R \to I^\bullet$ in* Mod-R, flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
2. *In a minimal injective resolution $R \to J^\bullet$ in* Mod-R^{op}, flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
3. *For any $1 \leq i \leq n+1$, any $X \in$ mod-R and any submodule M of* $\mathrm{Ext}^i_R(X,R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(M,R) = 0$ *for all* $0 \leq j < i$.
4. *For any $1 \leq i \leq n+1$, any $M \in$ mod-R^{op} and any submodule X of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(M,R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(X,R) = 0$ *for all* $0 \leq j < i$.

Definition 2 ([5]). Let R be a left and right noetherian ring. We say that R satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and that R is an Auslander-Gorenstein ring if it satisfies the Auslander condition

This is not in final form. The detailed version will be submitted for publication elsewhere.

and inj dim R^{op} = inj dim $R < \infty$. Also, an Auslander-Gorenstein ring R is said to be Auslander-regular if gl dim $R < \infty$.

Remark 3. Let R be a left and right noetherian ring. Assume dom dim $R \geq 2$, i.e., the first two terms I^0, I^1 in a minimal injective resolution $R \to I^\bullet$ in Mod-R are flat. It then follows by [7, Proposition 3.4] and [8, Corollary C] that R is left and right artinian.

Note that commutative Gorenstein rings are Auslander-Gorenstein (see [4]), and that if R is a left and right noetherian ring, and if inj dim $R^{\mathrm{op}} < \infty$ and inj dim $R < \infty$, then inj dim R^{op} = inj dim R (see e.g. [14, Lemma A]). Also, if R is a left and right noetherian ring then gl dim R^{op} = gl dim R.

Lemma 4. *For any $X, Y \in$ Mod-R we have a bifunctorial homomorphism*

$$\xi_{X,Y} : X \otimes_R \mathrm{Hom}_R(Y, R) \to \mathrm{Hom}_R(Y, X), x \otimes f \mapsto (y \mapsto xf(y))$$

and the following hold.

(1) *If either $X \in \mathcal{P}_R$ or $Y \in \mathcal{P}_R$ then $\xi_{X,Y}$ is an isomorphism.*
(2) *If $\xi_{X,X}$ is an epimorphism then $X \in \mathcal{P}_R$.*

Lemma 5 (Morita)**.** *Let A be an arbitrary ring. Then for any $V \in$ Mod-A, setting $B = \mathrm{End}_A(V)$ and $U = \mathrm{Hom}_A(V, A)$, the following hold.*

(1) *If $V_A \in \mathcal{P}_A$ then ${}_BB \in \mathrm{add}({}_BV)$.*
(2) *If $A_A \in \mathrm{add}(V_A)$ then ${}_BV \in \mathcal{P}_{B^{\mathrm{op}}}$ with $A \xrightarrow{\sim} \mathrm{End}_{B^{\mathrm{op}}}(V)^{\mathrm{op}}$ canonically.*
(3) *If $\mathrm{add}(V_A) = \mathcal{P}_A$ then $V \otimes_A U \cong B$ as B-bimodules and $U \otimes_B V \cong A$ as A-bimodules, so that we have equivalences*

$$V \otimes_A - : \text{Mod-}A^{\mathrm{op}} \xrightarrow{\sim} \text{Mod-}B^{\mathrm{op}} \text{ and } - \otimes_A U : \text{Mod-}A \xrightarrow{\sim} \text{Mod-}B.$$

Definition 6. Let A, B be rings. If there exists a module $V \in$ Mod-A such that $\mathrm{add}(V_A) = \mathcal{P}_A$ and $B \xrightarrow{\sim} \mathrm{End}_A(V)$, then B is said to be Morita equivalent to A. According to Lemma 5(3), B is Morita equivalent to A if and only if A is Morita euivalent to B. So, we say that A, B are Morita equivalent (to each other) if one is Morita equivalent to the other.

Lemma 7. *For any $X \in$ mod-R and any injective $E \in$ Mod-R^{op} we have a bifunctorial isomorphism*

$$\zeta_{X,E} : X \otimes_R E \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(X, R), E), x \otimes a \mapsto (f \mapsto f(x)a).$$

Recall that a projective resolution $P^\bullet \to X$ is said to be finite if the P^i are finitely generated.

Lemma 8. *Let $E \in$ Mod-R^{op} be injective. For any $X \in$ Mod-R with a finite projective resolution we have $\mathrm{Tor}_i^R(X, E) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Ext}_R^i(X, R), E)$ for all $i \geq 0$. In particular, if R is right noetherian, the following hold.*

(1) flat dim ${}_RE \leq$ inj dim R.
(2) *If ${}_RE$ is an injective cogenerator then* flat dim ${}_RE$ = inj dim R.

Lemma 9. *Let $\phi : R \to A$ be a ring homomorphism. Then for any $X \in$ Mod-A and $Y \in$ Mod-R we have a bifunctorial isomomorphism*

$$\eta_{X,Y} : \mathrm{Hom}_R(X, Y) \xrightarrow{\sim} \mathrm{Hom}_A(X, \mathrm{Hom}_R(A, Y)), f \mapsto (x \mapsto (a \mapsto f(xa))).$$

In particular, if $I \in \mathrm{Mod}\text{-}R$ *is injective (resp., an injective cogenerator) then so is* $\mathrm{Hom}_R(A, I) \in \mathrm{Mod}\text{-}A$.

Lemma 10. *Let* $\phi : R \to A$ *be a ring homomorphism with* $\mathrm{Ext}^i_R(A, R) = 0$ *for all* $i \geq 1$ *and set* $V = \mathrm{Hom}_R(A, R)$. *Then* inj dim $V_A \leq$ inj dim R, *where the equality holds if either* ϕ *is a split monomorphism of* R*-bimodules or* ϕ *is a split monomorphism in* $\mathrm{Mod}\text{-}R^{\mathrm{op}}$ *and* inj dim $R < \infty$.

Example 11. Let $R \to A$ be a ring homomorphism and set $\Gamma = \mathrm{End}_R(A)$ with $\psi : A \to \Gamma, a \mapsto (x \mapsto ax)$. Then $\varepsilon \circ \psi = \mathrm{id}_A$ with $\varepsilon : \Gamma \to A, \gamma \mapsto \gamma(1_A)$ and ψ is a split monomorphism of (A, R)-bimodules.

Definition 12. A module $X \in \mathrm{Mod}\text{-}R$ is said to be torsionless (resp., reflexive) if the evaluation map

$$\varepsilon_X : X \to \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(X, R), R), x \mapsto (f \mapsto f(x))$$

is a monomorphism (resp., an isomorphism).

Definition 13. A ring homomorphism $\phi : R \to A$ is said to be separable (resp., an epimorphism) if the multiplication map $\pi : A \otimes_R A \to A, a \otimes b \mapsto ab$ is a split epimorphism (resp., an isomorphism) of A-bimodules.

Example 14. Let $\Gamma = \mathrm{M}_2(R)$ be the ring of 2×2 full matrices over a ring R and $A = \mathrm{T}_2(R)$ the subring of Γ consisting of upper triangular matrices. Then the inclusion $A \to \Gamma$ is a ring epimorphism and the canonical ring homomorphism $R \to \Gamma, r \mapsto \mathrm{diag}(r, r)$ is separable.

Lemma 15. *Let* $\phi : R \to A$ *be a separable ring homomorphism. Assume either* ${}_RA$ *is flat or* A_R *is projective. Then for any* $X, Y \in \mathrm{Mod}\text{-}A$ *we have a bifunctorial split monomorphism* $\mathrm{Ext}^i_A(X, Y) \to \mathrm{Ext}^i_R(X, Y)$ *for all* $i \geq 0$ *and hence* gl dim $A \leq$ gl dim R. *If* $A_R \in \mathcal{P}_R$ *then* inj dim $A \leq$ inj dim R.

2. Frobenius ring homomorphisms

Throughout the rest of this note, we denote by $\mathcal{G}_R$ the full subcategory of mod-R consisting of $X \in$ mod-R admitting a finite projective resolution and with $\mathrm{Ext}^i_R(X, R) = 0$ for all $i \geq 1$. Obviously, we have $\mathcal{P}_R \subset \mathcal{G}_R \subset$ mod-R. Note also that if R is right noetherian then every finitely generated $X \in \mathrm{Mod}\text{-}R$ admits a finite projective resolution.

Throughout this section, we fix a ring homomorphism $\phi : R \to A$ and set $V = \mathrm{Hom}_R(A, R)$ which is an (R, A)-bimodule.

Lemma 16. *If* $A_R \in \mathcal{G}_R$ *then for any injective* $E \in \mathrm{Mod}\text{-}R^{\mathrm{op}}$ *the following hold.*

1. $\mathrm{Tor}^R_i(A, E) = 0$ *for all* $i \geq 1$.
2. flat dim ${}_AA \otimes_R E \leq$ flat dim ${}_RE$.
3. *If* V_A *is flat then* ${}_AA \otimes_R E$ *is injective.*
4. *Assume* V_A *is faithfully flat. If* ${}_RE$ *is an injective cogenerator then so is* ${}_AA \otimes_R E$.

Corollary 17. *If* $A_R \in \mathcal{G}_R$ *and* V_A *is flat then* inj dim ${}_AA \otimes_R M \leq$ inj dim ${}_RM$ *for all* $M \in \mathrm{Mod}\text{-}R^{\mathrm{op}}$.

Proposition 18 (cf. [9, Proposition 1.7(1)])**.** *Assume $A_R \in \mathcal{G}_R$, ${}_RA$ is finitely generated and V_A is faithfully flat. If R is Auslander-Gorenstein then so is A.*

In case A has been known to be left and right noetherian, in the proposition above we need not to assume ${}_RA$ is finitely generated.

As in Lemma 4, for any $X \in \text{Mod-}R$ we have a functorial homomorphism

$$\xi_X : X \otimes_R V \to \operatorname{Hom}_R(A, X), x \otimes v \mapsto (a \mapsto xv(a)).$$

Lemma 19. *If $A_R \in \mathcal{G}_R$ then for any $X \in \text{Mod-}R$ with* flat dim $X_R < \infty$ *the following hold.*

(1) $\operatorname{Tor}_i^R(X, V) = 0$ *for all $i \geq 1$.*
(2) *ξ_X is an isomorphism.*
(3) flat dim $\operatorname{Hom}_R(A, X)_A \leq$ flat dim $X_R +$ flat dim V_A.

Definition 20 (cf. [1] and [11, 12])**.** We call ϕ a Frobenius ring homomorphism if $A_R \in \mathcal{G}_R$ and $\operatorname{add}(V_A) = \mathcal{P}_A$ (cf. also Proposition 51 below). In case ϕ is injective, we identify R with $\phi(R) \subset A$ and call A a Frobenius extension of R.

Proposition 21. *Assume R is right noetherian and ϕ is Frobenius. Then a ring homomorphism $\psi : A \to \Gamma$ is Frobenius if and only if so is $\psi \circ \phi$.*

Remark 22. In the proposition above, the "only if" part holds without the assumption that R is right noetherian. Namely, if Γ_A admits a finite projective resolution $Q^\bullet \to \Gamma$ in Mod-A, and if every Q^i admits a finite projective resolution $P^{i\bullet} \to Q^i$ in Mod-R, then we have a double complex $P^{\bullet\bullet}$ over $\mathcal{P}_R$ the total complex of which yields a finite projective resolution of Γ_R.

Theorem 23. *Assume R is left and right noetherian. If ϕ is Frobenius then the following hold.*

(1) *A is left and right noetherian.*
(2) *If* inj dim $R^{\text{op}} =$ inj dim $R = d$ *then* inj dim $A^{\text{op}} =$ inj dim $A \leq d$.
(3) *If R satisfies the Auslander condition then so does A.*
(4) *If R is Auslander-Gorenstein then so is A.*

In the theorem above, we do not know whether or not ${}_RA$ is finitely generated. Also, it may happen that inj dim $A < d$ (see Example 26 below).

Throughout the rest of this section, we set $\Gamma = \operatorname{End}_{R^{\text{op}}}(A)^{\text{op}}$, which contains A as a subring via the injective ring homomorphism $A \to \Gamma, a \mapsto (x \mapsto xa)$, and set $U = \operatorname{Hom}_{R^{\text{op}}}(A, R)$ and $\Delta = \operatorname{Hom}_A(\Gamma, A)$.

Lemma 24. *If ${}_RA \in \mathcal{P}_{R^{\text{op}}}$ then $U_R \in \mathcal{P}_R$ and the following hold.*

(1) *$\Gamma \cong U \otimes_R A$ as Γ-bimodules. In particular, $\Gamma_A \in \mathcal{P}_A$.*
(2) *$\Delta \cong A \otimes_R A$ as (A, Γ)-bimodules.*
(3) *If $\operatorname{add}(U_R) = \operatorname{add}(A_R)$ then $\operatorname{add}(\Delta_\Gamma) = \mathcal{P}_\Gamma$ and hence Γ is a Frobenius extension of A.*
(4) *If $\operatorname{add}({}_RA) = \mathcal{P}_{R^{\text{op}}}$ then $\operatorname{add}(\Gamma_A) = \mathcal{P}_A$.*

Theorem 25. *Assume $\operatorname{add}({}_RA) = \mathcal{P}_{R^{\text{op}}}$ and $\operatorname{add}(A_R) = \mathcal{P}_R$. Then the following hold.*

(1) *If A is left and right noetherian then so is R.*
(2) *If A is Auslander-Gorenstein then so is R.*
(3) *If ϕ is Frobenius then* inj dim $A =$ inj dim R.

Example 26. Assume R is a commutative noetherian local ring and A is an R-algebra, i.e., Im ϕ is contained in the center of A, such that A is a free R-module of finite rank. Then Γ is a Frobenius extension of A and is Morita equivalent to R, so that if A is Auslander-Gorenstein then R has to be Gorenstein. However, even if A is Auslander-Gorenstein, it may happen that inj dim $A >$ inj dim $\Gamma = \dim R$. For instance, consider the case where $A = \mathrm{T}_2(R)$, the ring of 2×2 upper triangular matrices over R. Then A is an Auslander-Gorenstein ring with inj dim $A = \dim R + 1$. In fact, $V_A \notin \mathcal{P}_A$ and A is not a Frobenius extension of R.

3. Frobenius bimodules

In this section, we introduce the notion of Frobenius bimodules. If R is a subring of a ring A such that $A_R \in \mathcal{P}_R$ and $A \cong \mathrm{Hom}_R(A, R)$ as (R, A)-modules, then in [1] the ring extension A/R is said to be a Frobenius extension of first kind (cf. also [11, 12]). We will generalize this notion.

To begin with, we notice the following facts.

Proposition 27. *Let $\phi : R \to A$ be a Frobenius and separable ring homomorphism with $A_R \in \mathcal{P}_R$. If R is Auslander-regular then so is A.*

Proposition 28. *Let V be an (A, R)-bimodule with ${}_AV \in \mathcal{P}_{A^{\mathrm{op}}}$ and set $\Gamma = \mathrm{End}_{A^{\mathrm{op}}}(V)^{\mathrm{op}}$ with $R \to \Gamma, r \mapsto (v \mapsto vr)$. Then the following hold.*

(1) $\Gamma_R \in \mathrm{add}(V_R)$
(2) *If* $\mathrm{add}(\mathrm{Hom}_R(V, R)_A) = \mathrm{add}(\mathrm{Hom}_{A^{\mathrm{op}}}(V, A)_A)$ *then* $\mathrm{add}(\mathrm{Hom}_R(\Gamma, R)_\Gamma) = \mathcal{P}_\Gamma$.

Definition 29. An (A, R)-bimodule V is said to be Frobenius if $V_R \in \mathcal{P}_R$, ${}_AV \in \mathcal{P}_{A^{\mathrm{op}}}$ and $\mathrm{Hom}_R(V, R) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(V, A)$ as (R, A)-bimodules.

Example 30. Let $V \in$ Mod-A with $\mathrm{add}(V_A) = \mathcal{P}_A$ and set $B = \mathrm{End}_A(V)$. It then follows by Lemma 5 that V is a Frobenius (B, A)-bimodule.

Proposition 31. *Let V be an (A, R)-bimodule and Δ a (Γ, A)-bimodule. If both V and Δ are Frobenius then so is $\Delta \otimes_A V$.*

Lemma 32. *Let V be a Frobenius (A, R)-bimodule and set $\Gamma = \mathrm{End}_{A^{\mathrm{op}}}(V)^{\mathrm{op}}$ and $\psi : R \to \Gamma, r \mapsto (v \mapsto vr)$. Then the following hold.*

(1) $\mathrm{Hom}_R(V, R)$ *is a Frobenius (R, A)-bimodule.*
(2) *Γ is a Frobenius (Γ, R)-bimodule.*

Lemma 33. *For any ring homomorphism $\phi : R \to A$ the following are equivalent.*

(1) *A is a Frobenius (A, R)-bimodule.*
(2) *A is a Frobenius (R, A)-bimodule.*

Throughout the rest of this section, we fix a ring homomorphism $\phi : R \to A$ and set $V = \mathrm{Hom}_R(A, R)$, $\Gamma = \mathrm{End}_{R^{\mathrm{op}}}(A)^{\mathrm{op}}$ and $\psi : A \to \Gamma, a \mapsto (x \mapsto xa)$.

Proposition 34. *If $A_R \in \mathcal{P}_R$ and ${}_AA \cong {}_AV$ then we have $\rho : \mathrm{End}_R(A) \xrightarrow{\sim} \mathit{\Gamma}$ such that $\psi = \rho \circ \psi'$ with $\psi' : A \to \mathrm{End}_R(A), a \mapsto (x \mapsto ax)$.*

Theorem 35. *If A is a Frobenius (A, R)-bimodule then the following hold.*

(1) *ϕ is Frobenius.*
(2) *$\mathit{\Gamma}$ is a Frobenius $(\mathit{\Gamma}, A)$-bimodule.*
(3) *If $\mathrm{add}({}_RA) = \mathcal{P}_{R^{\mathrm{op}}}$ then $\mathrm{add}({}_A\mathit{\Gamma}) = \mathcal{P}_{A^{\mathrm{op}}}$.*

Corollary 36. *Let $\phi_0 : A_0 \to A_1$ be a ring homomorphism and set $A_{i+1} = \mathrm{End}_{A_{i-1}^{\mathrm{op}}}(A_i)^{\mathrm{op}}$ and $\phi_i : A_i \to A_{i+1}, a \mapsto (x \mapsto xa)$ for $i \geq 1$ inductively. If A_1 is a Frobenius (A_1, A_0)-bimodule then A_i is a Frobenius (A_i, A_{i-1})-bimodule for all $i \geq 1$.*

In the following, we denote by $\pi : A \otimes_R A \to A, a \otimes b \mapsto ab$ the multiplication map. Note that ϕ is separable if and only if there exists $\delta \in A \otimes_R A$ such that $\pi(\delta) = 1_A$ and $a\delta = \delta a$ for all $a \in A$.

Proposition 37. *The following hold.*

(1) *If ϕ is separable then ψ is a split monomorphism of A-bimodules.*
(2) *If ${}_AA \otimes_R A$ is reflexive, and if ψ is a split monomorphism of A-bimodules, then ϕ is separable.*

Lemma 38. *Assume A is a Frobenius (A, R)-bimodule with $\varphi : A \xrightarrow{\sim} V$ an isomorphism of (R, A)-bimodules. Then $\tau = \varphi(1_A) : A \to R$ is a homomorphism of R-bimodules and the following hold.*

(1) *$\varphi' : A \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(A, R), a \mapsto a\tau$ as (A, R)-bimodules.*
(2) *$\xi : A \otimes_R A \xrightarrow{\sim} \mathit{\Gamma}, a \otimes b \mapsto (x \mapsto \tau(xa)b)$ as A-bimodules.*
(3) *$\xi' : A \otimes_R A \xrightarrow{\sim} \mathrm{End}_R(A), a \otimes b \mapsto (x \mapsto a\tau(bx))$ as A-bimodules.*
(4) *$\xi = \rho \circ \xi'$.*

Theorem 39. *Assume A is a Frobenius (A, R)-bimodule. Let $\varphi : A \xrightarrow{\sim} V$ be an isomorphism of (R, A)-bimodules and assume $\tau = \varphi(1_A) : A \to R$ is a split epimorphism of R-bimodules. Then the following hold.*

(1) *$\mathit{\Gamma}$ is Morita equivalent to R.*
(2) *ψ is separable.*
(3) *If A is Auslander-regular then so is R.*

Example 40. Consider the case where R is a commutative field and $A = R[t]/(t^2)$. Then, setting $\tau : A \to R, r_0 + r_1 t \mapsto r_1$, we have $A \xrightarrow{\sim} V, r \mapsto r\tau$ as (R, A)-bimodules. Thus A is a Frobenius (A, R)-bimodule. Since the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_R A \cong R[x, y]/(x^2, y^2)$ is local, and since $\dim_R A^e \neq \dim_R A$, π does not split in Mod-A^e, i.e., ϕ is not separable. On the other hand, since τ is a split epimorphism of R-bimodules, ψ is separable.

4. Double centralizer

Throughout this section, we fix a finitely generated projective module $P \in \mathcal{P}_R$ and set $A = \mathrm{End}_R(P)$ and $\mathit{\Gamma} = \mathrm{End}_{A^{\mathrm{op}}}(P)^{\mathrm{op}}$ with $\phi : R \to \mathit{\Gamma}, r \mapsto (x \mapsto xr)$ the canonical ring homomorphism. We will provide a sufficient condition for ϕ to be a Frobenius ring epimorphism with $\Gamma_R \in \mathcal{P}_R$. We refer to [13] for general theory of localization in module categories.

In the following, we set $Q = \mathrm{Hom}_R(P, R)$ and $V = Q \otimes_A P$. Note that by Lemma 5(1) ${}_AA \in \mathrm{add}({}_AP)$ and $A_A \in \mathrm{add}(Q_A)$ and that by Lemma 5(2) $P_\Gamma \in \mathcal{P}_\Gamma$ with $A \xrightarrow{\sim} \mathrm{End}_\Gamma(P)$ canonically. Also, by Lemma 4(1) $\mathrm{Hom}_R(P, -) \cong - \otimes_R Q$ and $\mathrm{Hom}_{R^{\mathrm{op}}}(Q, -) \cong P \otimes_R -$.

Lemma 41. *The following hold.*

(1) ${}_AA \in \mathrm{add}({}_AP)$ *and* $P_\Gamma \in \mathcal{P}_\Gamma$ *with* $A \xrightarrow{\sim} \mathrm{End}_\Gamma(P)$ *canonically, so that* $\Gamma_\Gamma \in \mathrm{add}(P_\Gamma)$ *if and only if* ${}_AP \in \mathcal{P}_{A^{\mathrm{op}}}$.

(2) ${}_RQ \in \mathcal{P}_{R^{\mathrm{op}}}$ *with* $P_R \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(Q, R)_R, x \mapsto (f \mapsto f(x))$ *and* $A \xrightarrow{\sim} \mathrm{End}_{R^{\mathrm{op}}}(Q)^{\mathrm{op}}$ *canonically, so that* $A_A \in \mathrm{add}(Q_A)$ *and* $P_\Gamma \in \mathrm{add}(V_\Gamma)$.

(3) $P \otimes_R Q \cong A$ *as A-bimodules, so that* $P \otimes_R V \cong P$ *as (A, Γ)-bimodules and* $V \otimes_R Q \cong Q$ *as (R, A)-bimodules. In particular,* $V \otimes_R V \cong V$ *as (R, Γ)-bimodules.*

(4) $\mathrm{Hom}_{R^{\mathrm{op}}}(V, R) \cong \Gamma$ *as (Γ, R)-bimodules, so that* ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *if and only if* $\Gamma_R \in \mathcal{P}_R$ *with* $V \xrightarrow{\sim} \mathrm{Hom}_R(\Gamma, R)$ *as (R, Γ)-bimodules.*

Lemma 42. *The following hold.*

(1) $(P \otimes_R -) \circ (Q \otimes_A -) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(Q, -) \circ \mathrm{Hom}_{A^{\mathrm{op}}}(P, -) \cong \mathbf{1}_{\text{Mod-}A^{\mathrm{op}}}$, *so that both* ${}_RQ \otimes_A -$ *and* $\mathrm{Hom}_{A^{\mathrm{op}}}({}_AP_R, -)$ *are fully faithful.*

(2) $\Gamma \xrightarrow{\sim} \mathrm{End}_{R^{\mathrm{op}}}(V)^{\mathrm{op}}$ *canonically, so that if* ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *then* $\Gamma_\Gamma \in \mathrm{add}(V_\Gamma)$.

(3) $\Gamma \xrightarrow{\sim} \mathrm{End}_{R^{\mathrm{op}}}(\Gamma)^{\mathrm{op}}$ *canonically, so that if* ${}_RR \in \mathrm{add}({}_R\Gamma)$ *then* $\phi : R \xrightarrow{\sim} \Gamma$.

Proposition 43. *If* ${}_\Gamma\Gamma \otimes_R \Gamma$ *is torsionless then* ϕ *is a ring epimorphism.*

Lemma 44. *The following hold.*

(1) *For any* $M \in \text{Mod-}R^{\mathrm{op}}$ *we have a functorial homomorphism in* $\text{Mod-}\Gamma^{\mathrm{op}}$

$$\omega_M : \Gamma \otimes_R M \to \mathrm{Hom}_{A^{\mathrm{op}}}(P, P \otimes_R M), \gamma \otimes m \mapsto (x \mapsto x\gamma \otimes m)$$

which is an isomorphism if either ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *or* ${}_RM \in \mathcal{P}_{R^{\mathrm{op}}}$.

(2) $P \otimes_R \Gamma \xrightarrow{\sim} P, x \otimes \gamma \mapsto x\gamma$, *so that* $\mathrm{Hom}_\Gamma(P, -) \cong \mathrm{Hom}_R(P, -)$ *on* $\text{Mod-}\Gamma$ *and* $P \otimes_R - \cong P \otimes_\Gamma -$ *on* $\text{Mod-}\Gamma^{\mathrm{op}}$. *In particular, if either* ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *or* ${}_R\Gamma \in \mathcal{P}_{R^{\mathrm{op}}}$ *then ϕ is a ring epimorphism.*

(3) $\mathrm{Hom}_{A^{\mathrm{op}}}(P, A) \cong \mathrm{Hom}_\Gamma(P, \Gamma) \cong \Gamma \otimes_R Q$ *as (Γ, A)-bimodules, so that if* ${}_AP \in \mathcal{P}_{A^{\mathrm{op}}}$ *then P is a Frobenius (A, Γ)-bimodule.*

Theorem 45. *If* ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *then the following hold.*

(1) ϕ *is a ring epimorphism with* $\Gamma_R \in \mathcal{P}_R$, *so that* $\text{inj dim } \Gamma \leq \text{inj dim } R$ *and* $\text{gl dim } \Gamma \leq \text{gl dim } R$.

(2) *If* $V_\Gamma \in \mathcal{P}_\Gamma$ *then ϕ is Frobenius.*

Lemma 46. *Let* $\varepsilon : V \to R, f \otimes x \mapsto f(x)$ *and* $\mathfrak{a} = \mathrm{Im}\ \varepsilon$. *Then the following hold.*

(1) $\mathfrak{a}^2 = \mathfrak{a}$ *and* $\mathrm{Ker}(- \otimes_R Q) = \text{Mod-}(R/\mathfrak{a})$.

(2) ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *if and only if* ${}_AP \in \mathcal{P}_{A^{\mathrm{op}}}$.

Lemma 47. *If* ${}_AP \in \mathcal{P}_{A^{\mathrm{op}}}$ *then the following hold.*

(1) $\mathrm{Hom}_R({}_\Gamma\Gamma_R, -) \cong \mathrm{Hom}_R({}_AP_R, -) \otimes_A P_\Gamma$.

(2) $- \otimes_A P_\Gamma : \text{Mod-}A \xrightarrow{\sim} \text{Mod-}\Gamma$.

(3) $V_\Gamma \in \mathcal{P}_\Gamma$ *if and only if* $Q_A \in \mathcal{P}_A$.

Proposition 48. *The following are equivalent.*

(1) ${}_RV \in \mathcal{P}_{R^{\mathrm{op}}}$ *and* $V_\Gamma \in \mathcal{P}_\Gamma$.
(2) ${}_AP \in \mathcal{P}_{A^{\mathrm{op}}}$ *and* $Q_A \in \mathcal{P}_A$.

Example 49. (1) Let V be an (A, B)-bimodule with ${}_AV \in \mathcal{P}_{A^{\mathrm{op}}}$ and set

$$R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, setting $P = eR$, we have $A \cong eRe$ and $Q \cong Re$, so that ${}_AP \cong {}_AA \oplus {}_AV \in \mathcal{P}_{A^{\mathrm{op}}}$ and $Q_A \cong A_A \in \mathcal{P}_A$.

(2) Let A be a commutative ring and R an A-algebra with $R_A \in \mathcal{P}_A$. Assume R contains an idempotent e with $eRe = Ae$. Then, setting $P = eR$, we have $A \cong eRe$ and $Q \cong Re$, so that ${}_AP \in \mathcal{P}_{A^{\mathrm{op}}}$ and $Q_A \in \mathcal{P}_A$.

5. Gorenstein projectives

In this final section, we deal with some questions in homological algebra which are still open. We denote by $\mathcal{G}_R^0$ the full subcategory of $\mathcal{G}_R$ consisting of $X \in \mathcal{G}_R$ with $\mathrm{Hom}_R(X, R) = 0$. The generalized Nakayama conjecture asserts that if R is right noetherian then $\mathcal{G}_R^0$ would contain no simple module (see [3] for details).

To begin with, we recall the notion of Gorenstein projective modules.

Definition 50 ([6])**.** A module $X \in \text{mod-}R$ is said to be Gorenstein projective if it is reflexive with $X \in \mathcal{G}_R$ and $\mathrm{Hom}_R(X, R) \in \mathcal{G}_{R^{\mathrm{op}}}$, i.e., there exists a complex $P^\bullet$ over $\mathcal{P}_R$ such that $\mathrm{Z}'^0(P^\bullet) \cong X$ and $\mathrm{H}^i(P^\bullet) = \mathrm{H}^{-i}(\mathrm{Hom}_R^\bullet(P^\bullet, R)) = 0$ for all $i \in \mathbb{Z}$.

It is obvious that if X_R is reflexive (resp., Gorenstein projective) then so is ${}_R\mathrm{Hom}_R(X, R)$. Thus the notion of Frobenius ring homomorphisms could be slightly modified to be symmetric in the following sense.

Proposition 51. *Let* $\phi : R \to A$ *be a ring homomorphism and set* $V = \mathrm{Hom}_R(A, R)$, $B = \mathrm{End}_A(V)$ *and* $\psi : R \to B, r \mapsto (v \mapsto rv)$. *Then the following hold.*

(1) *If* A_R *is reflexive (resp., Gorenstein projective) and* $V_A \in \mathcal{P}_A$ *then* ${}_RB$ *is reflexive (resp., Gorenstein projective) and* $\mathrm{Hom}_{R^{\mathrm{op}}}(B, R) \cong V$ *as* (B, R)*-bimodules.*
(2) *If* $\mathrm{add}(V_A) = \mathcal{P}_A$ *then* $A \xrightarrow{\sim} \mathrm{End}_{B^{\mathrm{op}}}(V)^{\mathrm{op}}$ *canonically and* $\mathrm{add}({}_BV) = \mathcal{P}_{B^{\mathrm{op}}}$.
(3) *If* $A \cong V$ *as* (R, A)*-bimodules then there exists a ring isomorphism* $\sigma : A \xrightarrow{\sim} B$ *such that* $\psi = \sigma \circ \phi$.

Throughout the rest of this section, we fix a complete set of non-isomorphic simple modules $\{S_\lambda\}_{\lambda \in \Lambda}$ in $\text{Mod-}R^{\mathrm{op}}$ and for each $\lambda \in \Lambda$ we denote by $E_\lambda = E_{R^{\mathrm{op}}}(S_\lambda)$ the injective envelope of S_λ in $\text{Mod-}R^{\mathrm{op}}$.

Lemma 52. *For any* $M \in \text{Mod-}R^{\mathrm{op}}$ *with* $\mathrm{Hom}_{R^{\mathrm{op}}}(M, E_\lambda) = 0$ *for all* $\lambda \in \Lambda$ *we have* $M = 0$*, i.e.,* $\Pi_{\lambda \in \Lambda} E_\lambda \in \text{Mod-}R^{\mathrm{op}}$ *is an injective cogenerator.*

Corollary 53. *For any* $X \in \text{mod-}R$ *the following hold.*

(1) $\mathrm{Hom}_R(X, R) = 0$ *if and only if* $X \otimes_R E_\lambda = 0$ *for all* $\lambda \in \Lambda$.

(2) *If R is right noetherian then, for any $i \geq 0$, $\mathrm{Ext}^i_R(X,R) = 0$ if and only if $\mathrm{Tor}^R_i(X, E_\lambda) = 0$ for all $\lambda \in \Lambda$.*

Lemma 54 ([10, Corollary A.2])**.** *Assume R is left and right noetherian. If every $X \in \mathcal{G}_R$ is torsionless then $\mathcal{G}_R$ consists only of Gorenstein projectives.*

Throughout the rest of this section, we fix a ring homomorphism $\phi : R \to A$ and set $V = \mathrm{Hom}_R(A, R)$.

Proposition 55. *If $A_R \in \text{mod-}R$ then we have*

$$\text{flat dim } V_A = \sup\{\text{inj dim } {}_A A \otimes_R E_\lambda \mid \lambda \in \Lambda\}.$$

Example 56. If $A = \mathrm{T}_2(R)$, the ring of 2×2 upper triangular matrices over R, and $\phi : R \to A, r \mapsto \mathrm{diag}(r, r)$, then $A_R \in \mathcal{P}_R$ and proj dim $V_A = 1$.

In the following, we assume $A_R \in \mathcal{G}_R$ and inj dim ${}_A A \otimes_R E_\lambda < \infty$ for all $\lambda \in \Lambda$. In addition, to ensure that every finitely generated $X \in \text{Mod-}R$ admits a finite projective resolution, we assume R is right noetherian. Note that A also is right noetherian.

Lemma 57. *For any $X \in \mathcal{G}_A$ the following hold.*

(1) *$X_R \in \mathcal{G}_R$.*
(2) *If $X_A \in \mathcal{G}^0_A$ then $X_R \in \mathcal{G}^0_R$.*
(3) *If X_R is torsionless then so is X_A.*

Theorem 58. *The following hold.*

(1) *If $\mathcal{G}^0_R = \{0\}$ then $\mathcal{G}^0_A = \{0\}$.*
(2) *Assume X_R is semisimple for all simple $X \in \text{Mod-}A$. If the generalized Nakayama conjecture holds true for R then so does for A.*
(3) *Assume both R and A are left and right noetherian. If $\mathcal{G}_R$ consists only of Gorenstein projectives then so does $\mathcal{G}_A$.*

Remark 59 ([2]). Let $X \in \text{mod-}R$ with $P^\bullet \to X$ a finite projective resolution and set $M_n = \mathrm{Z}'^n(\mathrm{Hom}^\bullet_R(P^\bullet, R))$ for $n \geq 1$. Then for any $n \geq 1$, if $\mathrm{Ext}^i_R(X,R) = 0$ for $1 \leq i < n$, the following hold.

(1) $\mathrm{Ext}^i_{R^{\mathrm{op}}}(M_n, R) = 0$ for $1 \leq i < n$ and $\mathrm{Ext}^n_{R^{\mathrm{op}}}(M_n, R) \cong \mathrm{Ext}^1_{R^{\mathrm{op}}}(M_1, R)$.
(2) If $\mathrm{Hom}_R(X,R) = 0$ then $X \cong \mathrm{Ext}^n_{R^{\mathrm{op}}}(M_n, R)$ with proj dim $M_n \leq n$.
(3) X is torsionless if and only if $\mathrm{Ext}^n_{R^{\mathrm{op}}}(M_n, R) = 0$.

Remark 60. (1) J.-I. Miyachi has pointed out that in general $\mathcal{G}_R$ may contain a module which is neither simple nor torsionless (see [10, Example A.3]).

(2) For a minimal injective resolution $R \to E^\bullet$ in Mod-R, it is possible that $\oplus_{k=0}^\infty E^k$ is an injective cogenerator but $\oplus_{k=0}^n E^k$ is not for any $n \geq 0$. For instance, if R is a commutative Gorenstein ring of infinite dimension, then for any $n \geq 0$ there exists a maximal ideal $\mathfrak{m}$ of height $d > n$ and we have $\mathrm{Ext}^i_R(R/\mathfrak{m}, R) = 0$ for $0 \leq i < d$ and $\mathrm{Ext}^d_R(R/\mathfrak{m}, R) \neq 0$ (see [4] for details).

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